

Sum formulas of multiple zeta values with arguments are multiples of a positive integer

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Abstract

For $k \leq n$, let $E(mn, k)$ be the sum of all multiple zeta values of depth k and weight mn with arguments are multiples of $m \geq 2$. More precisely, $E(mn, k) = \sum_{|\alpha|=n} \zeta(m\alpha_1, m\alpha_2, \dots, m\alpha_k)$. In this paper, we develop a formula to express $E(mn, k)$ in terms of $\zeta(\{m\}^p)$ and $\zeta^*(\{m\}^q)$, $0 \leq p, q \leq n$. In particular, we settle Genčev's conjecture on the evaluation of $E(4n, k)$ and also evaluate $E(mn, k)$ explicitly for small even $m \leq 8$.

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1 Introduction

The multiple zeta values (MZVs) and the multiple zeta star values (MZSVs) are defined by [3, 6, 7, 9]

$$\begin{aligned}\zeta(\alpha_1, \alpha_2, \dots, \alpha_r) &= \sum_{1 \leq k_1 < k_2 < \dots < k_r} k_1^{-\alpha_1} k_2^{-\alpha_2} \dots k_r^{-\alpha_r} \quad \text{and} \\ \zeta^*(\alpha_1, \alpha_2, \dots, \alpha_r) &= \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_r} k_1^{-\alpha_1} k_2^{-\alpha_2} \dots k_r^{-\alpha_r}\end{aligned}$$

with positive integers $\alpha_1, \alpha_2, \dots, \alpha_r$ and $\alpha_r \geq 2$ for the sake of convergence. The numbers r and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_r$ are the depth and weight of $\zeta(\alpha)$ and $\zeta^*(\alpha)$. For our convenience, we let $\{a\}^k$ be k repetitions of a such that $\zeta(\{2\}^3) = \zeta(2, 2, 2)$ and $\zeta^*(\{3\}^4, 5) = \zeta^*(3, 3, 3, 3, 5)$.

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MZVs and MZSVs are strongly connected with each other, for example,

$$\begin{aligned}\zeta^*(s_1, s_2) &= \zeta(s_1, s_2) + \zeta(s_1 + s_2) \quad \text{and} \\ \zeta^*(s_1, s_2, s_3) &= \zeta(s_1, s_2, s_3) + \zeta(s_1 + s_2, s_3) + \zeta(s_1, s_2 + s_3) + \zeta(s_1 + s_2 + s_3).\end{aligned}$$

A principal goal in the theoretical study of MZVs or MZSVs is to determine all possible algebraic relations among them. Several explicit values are interesting and known for special index sets (e.g. [2, 3, 13, 17]). For example, Zagier [17] evaluated the value of $\zeta(\{2\}^a, 3, \{2\}^b)$. For a partition $\mathcal{P} = \{P_1, P_2, \dots, P_\ell\}$ of the set $[n] = \{1, 2, \dots, n\}$, let $c(\mathcal{P}) = (\text{card } P_1 - 1)!(\text{card } P_2 - 1)! \cdots (\text{card } P_\ell - 1)!$. Given a n -tuple $\mathbf{i} = \{i_1, i_2, \dots, i_n\}$, we define

$$\zeta(\mathbf{i}, \mathcal{P}) = \prod_{s=1}^{\ell} \zeta\left(\sum_{j \in P_s} i_j\right).$$

Let S_n be the symmetric group of degree n . In 1992, Hoffman [10] gave the evaluations of $\sum_{\sigma \in S_n} \zeta(i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)})$ and $\sum_{\sigma \in S_n} \zeta^*(i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)})$.

Proposition 1 ([10], Theorem 2.1, 2.2). *For any real $i_1, i_2, \dots, i_n > 1$, we have*

$$\sum_{\sigma \in S_n} \zeta(i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)}) = \sum_{\forall \mathcal{P} = \{P_1, \dots, P_\ell\} \text{ of } [n]} (-1)^{n-\ell} c(\mathcal{P}) \zeta(\mathbf{i}, \mathcal{P}), \quad (1)$$

$$\sum_{\sigma \in S_n} \zeta^*(i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n)}) = \sum_{\forall \mathcal{P} = \{P_1, \dots, P_\ell\} \text{ of } [n]} c(\mathcal{P}) \zeta(\mathbf{i}, \mathcal{P}). \quad (2)$$

It is known that for $m \geq 2$, the infinite product

$$\begin{aligned}\prod_{n=1}^{\infty} \left(1 + \frac{x^m}{n^m}\right) &= 1 + \sum_{k=1}^{\infty} \frac{1}{k^m} x^m + \sum_{1 \leq k_1 < k_2} \frac{1}{k_1^m k_2^m} x^{2m} + \cdots \\ &+ \sum_{1 \leq k_1 < k_2 < \cdots < k_n} \frac{1}{k_1^m k_2^m \cdots k_n^m} x^{mn} + \cdots.\end{aligned}$$

is the generating function of the sequence

$$1, \zeta(m), \zeta(m, m), \dots, \zeta(\{m\}^n), \dots$$

On the other hand, the generating function of $\zeta^*(\{m\}^n)$ [11, 13] is given by

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^m}{n^m}\right)^{-1} = \prod_{n=1}^{\infty} \left(1 + \frac{x^m}{n^m} + \frac{x^{2m}}{n^{2m}} + \cdots + \frac{x^{km}}{n^{km}} + \cdots\right),$$

so that $\zeta^*(\{m\}^n)$ has another expression

$$\sum_{k=1}^n \sum_{|\alpha|=n} \zeta(m\alpha_1, m\alpha_2, \dots, m\alpha_k).$$

Let $E(mn, k)$ be the sum of all multiple zeta values of depth k and weight mn with arguments are multiples of $m \geq 2$. More precisely,

$$E(mn, k) = \sum_{|\alpha|=n} \zeta(m\alpha_1, m\alpha_2, \dots, m\alpha_k).$$

This sum of multiple zeta values $E(mn, k)$ is just a part of $\zeta^*(\{m\}^n)$.

Gangl, Kaneko, and Zagier [8] proved that $E(2n, 2) = \frac{3}{4}\zeta(2n)$, for $n \geq 2$. Shen and Cai [14] gave formulas for $E(2n, 3)$ and $E(2n, 4)$ in terms of $\zeta(2n)$ and $\zeta(2)\zeta(2n-2)$. Using an explicit generating function for $E(2n, k)$, Hoffman [11] gave a general formula of $E(2n, k)$ for $k \leq n$.

Here we are able to express $E(mn, k)$ in terms of $\zeta(\{m\}^p)$ and $\zeta^*(\{m\}^q)$ with $p + q = n$ and $p \geq k$. By convention we let $\zeta(\{m\}^0) = 1$ and $\zeta^*(\{m\}^0) = 1$.

Theorem A. *For a pair of positive integers m, n with $m \geq 2$, we have for $1 \leq k \leq n$,*

$$E(mn, k) = \sum_{p+q=n} (-1)^{p-k} \binom{p}{k} \zeta(\{m\}^p) \zeta^*(\{m\}^q).$$

By inputing the explicit values of $\zeta(\{m\}^p)$ and $\zeta^*(\{m\}^q)$ for various m , we obtain the sum $E(mn, k)$. In particular, the following are most interesting so we list them as theorems.

Theorem B. *For a pair of positive integers n, k with $1 \leq k \leq n$, we have*

$$\begin{aligned} E(2n, k) &= \sum_{|\alpha|=n} \zeta(2\alpha_1, 2\alpha_2, \dots, 2\alpha_k) \\ &= \frac{(-1)^{n-k} \pi^{2n}}{(2n+1)!} \sum_{q=0}^{n-k} \binom{n-q}{k} \binom{2n+1}{2q} 2^{2q} B_{2q} \left(\frac{1}{2}\right). \end{aligned}$$

From the general identity [6, 7, 16]

$$B_n(kx) = k^{n-1} \sum_{j=0}^{k-1} B_n \left(x + \frac{j}{k}\right),$$

we have

$$2^{2q} B_{2q} \left(\frac{1}{2}\right) = (2 - 2^{2q}) B_{2q}.$$

So our Theorem B coincides with the result of Hoffman [11]

$$E(2n, k) = \frac{(-1)^{n-k-1} \pi^{2n}}{(2n+1)!} \sum_{j=0}^{n-k} \binom{n-j}{k} \binom{2n+1}{2j} 2(2^{2j-1} - 1) B_{2j}.$$

The evaluation of $E(4n, k)$ appeared in Genčev's paper [9] as a conjecture. Here we confirm the conjecture is true.

Theorem C. For a pair of positive integers n, k with $1 \leq k \leq n$, we have

$$E(4n, k) = \frac{2^{2n+1} \pi^{4n}}{(4n+2)!} \sum_{q=0}^{n-k} (-1)^{n-q-k} \binom{n-q}{k} \binom{4n+2}{4q} \sum_{|q|=2q} \binom{4q}{2q_1} (-1)^{q_1} 2^{2q} B_{2q_1} \left(\frac{1}{2} \right) B_{2q_2} \left(\frac{1}{2} \right).$$

Our purpose of this paper is to enable the evaluation of $E(2mn, k)$ directly by the fact that the evaluations of $\zeta(\{2m\}^p)$ and $\zeta^*(\{2m\}^q)$ are known. This paper is organized as follows. In Section 2, we present proofs of Theorem A and Theorem B. In Section 3 and 4, we rewrite several evaluations of $\zeta(\{2m\}^n)$ and $\zeta^*(\{2m\}^n)$ in the literature by dividing into two cases according to m is even or odd. In addition, we give a different expression of $E(2mn, k)$ in the final section.

2 Proofs of Theorem A and Theorem B

In order to evaluate the special values at even integers of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1,$$

Euler developed the infinite product formula of the sine function

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right).$$

The infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right)$$

is the generating function of $(-1)^n \zeta(\{2\}^n)$. Indeed we have

$$\begin{aligned} \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2} \right) &= 1 - \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) x^2 + \left(\sum_{1 \leq k_1 < k_2} \frac{1}{k_1^2 k_2^2} \right) x^4 + \cdots \\ &\quad + (-1)^n \left(\sum_{1 \leq k_1 < k_2 < \cdots < k_n} \frac{1}{k_1^2 k_2^2 \cdots k_n^2} \right) x^{2n} + \cdots \\ &= 1 - \zeta(2)x^2 + \zeta(2, 2)x^4 + \cdots + (-1)^n \zeta(\{2\}^n)x^{2n} + \cdots. \end{aligned}$$

Using the power series expansion

$$\frac{\sin \pi x}{\pi x} = 1 - \frac{(\pi x)^2}{3!} + \frac{(\pi x)^4}{5!} + \cdots + \frac{(-1)^n (\pi x)^{2n}}{(2n+1)!} + \cdots,$$

then it implies that the evaluation

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}.$$

On the other hand, the inverse of the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)^{-1} = \prod_{n=1}^{\infty} \left\{1 + \frac{x^2}{n^2} + \left(\frac{x^2}{n^2}\right)^2 + \cdots + \left(\frac{x^2}{n^2}\right)^k + \cdots\right\}$$

is the generating function of the sequence of sum of multiple zeta values [11]

$$\zeta^*(\{2\}^n) = \sum_{k=1}^n E(2n, k) = \sum_{k=1}^n \sum_{|\alpha|=n} \zeta(2\alpha_1, 2\alpha_2, \dots, 2\alpha_k).$$

Also such an infinite product, according to the infinite product formula of the sine function, is equal to

$$\frac{\pi x}{\sin \pi x}$$

or

$$\frac{2\pi i x e^{\pi i x}}{e^{2\pi i x} - 1}$$

and has the power series expansion

$$\sum_{n=0}^{\infty} \frac{(2\pi i x)^{2n}}{(2n)!} B_{2n} \left(\frac{1}{2}\right),$$

where $B_n(x)$ are Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{n!} \quad \text{for } n = 0, 1, 2, \dots$$

Here we summarize our previous discussion as follows which we needed in the evaluation of $E(2n, k)$.

Proposition 2 ([11]). *For any positive integer n , we have*

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}, \tag{3}$$

$$\zeta^*(\{2\}^n) = \sum_{k=1}^n \sum_{|\alpha|=n} \zeta(2\alpha_1, 2\alpha_2, \dots, 2\alpha_k) = (2\pi i)^{2n} \frac{B_{2n}(\frac{1}{2})}{(2n)!} = \frac{(-1)^n \pi^{2n}}{(2n)!} (2 - 2^{2n}) B_{2n}. \tag{4}$$

Now we are ready to prove Theorem A and Theorem B.

Proof of Theorem A. For real number λ , we consider the infinite product

$$F_m(\lambda, x) = \prod_{n=1}^{\infty} \left(1 + \frac{\lambda x^m}{n^m}\right) \left(1 - \frac{x^m}{n^m}\right)^{-1}$$

which is a product of two infinite products

$$\prod_{n=1}^{\infty} \left(1 + \frac{\lambda x^m}{n^m}\right) \quad \text{and} \quad \prod_{n=1}^{\infty} \left(1 - \frac{x^m}{n^m}\right)^{-1}.$$

The above products are generating functions of

$$\lambda^n \zeta(\{m\}^n) \quad \text{and} \quad \zeta^*(\{m\}^n),$$

respectively. Therefore, $F_m(\lambda, x)$ is the generating function of the convolution of the two corresponding sequences. Hence the coefficient of x^{mn} of $F_m(\lambda, x)$ is given by

$$\sum_{p+q=n} \lambda^p \zeta(\{m\}^p) \zeta^*(\{m\}^q).$$

On the other hand, we rewrite $F_m(\lambda, x)$ as

$$F_m(\lambda, x) = \prod_{n=1}^{\infty} \left\{ 1 + (\lambda + 1) \frac{x^m}{n^m} + (\lambda + 1) \frac{x^{2m}}{n^{2m}} + \cdots + (\lambda + 1) \frac{x^{km}}{n^{km}} + \cdots \right\}$$

and its coefficient of x^{mn} is given by

$$\sum_{r=1}^n (\lambda + 1)^r E(mn, r).$$

This leads to the identity

$$\sum_{r=1}^n (\lambda + 1)^r E(mn, r) = \sum_{p+q=n} \lambda^p \zeta(\{m\}^p) \zeta^*(\{m\}^q). \quad (5)$$

Differentiate both sides of the above identity with respect to λ for k times and then set $\lambda = -1$, we obtain that

$$E(mn, k) = \sum_{p+q=n} (-1)^{p-k} \binom{p}{k} \zeta(\{m\}^p) \zeta^*(\{m\}^q).$$

□

Taking $\lambda = 1$ into Eq.(5), we have the following.

Corollary 1. For a pair of positive integers m, n with $m \geq 2$, we have for $1 \leq k \leq n$,

$$\sum_{r=1}^n 2^r E(mn, r) = \sum_{p+q=n} \zeta(\{m\}^p) \zeta^*(\{m\}^q).$$

Proof of Theorem B. Substitute $\zeta(\{2\}^p)$ and $\zeta^*(\{2\}^q)$ by Eq.(3) and Eq.(4) into Theorem A, we obtain that

$$\begin{aligned} E(2n, k) &= \sum_{p+q=n} (-1)^{p-k} \binom{p}{k} \frac{\pi^{2p}}{(2p+1)!} (2\pi i)^{2q} \frac{B_{2q}(\frac{1}{2})}{(2q)!} \\ &= \frac{(-1)^{n-k} \pi^{2n}}{(2n+1)!} \sum_{q=0}^{n-k} \binom{n-q}{k} \binom{2n+1}{2q} 2^{2q} B_{2q} \left(\frac{1}{2} \right). \end{aligned}$$

□

3 Evaluations of $\zeta(\{2m\}^n)$

The evaluation of $\zeta(\{2m\}^n)$ is available in [1, 13]. However, what we need is the explicit value of $\zeta(\{2m\}^n)$. So we calculate them directly here for the reason of self-content.

Theorem 1. Suppose that m and n are positive integers with $m \geq 3$ odd. Let $w = e^{2\pi i/m}$, then we have

$$\zeta(\{2m\}^n) = \frac{2 \cdot (2\pi)^{2mn}}{(2mn+m)!} \left\{ \sum_{r=1}^{(m-1)/2} (-1)^{r-1} \sum_{0 \leq j_1 < j_2 < \dots < j_r \leq m-1} (w^{j_1} + w^{j_2} + \dots + w^{j_r})^{2mn+m} \right\}.$$

Proof. The infinite product formula of the sine function implies that for $|x| < 1$ and $x \neq 0$,

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} (-1)^n \zeta(\{2m\}^n) x^{2mn} &= \prod_{n=1}^{\infty} \left(1 - \frac{x^{2m}}{n^{2m}} \right) \\ &= \prod_{n=1}^{\infty} \prod_{j=0}^{m-1} \left(1 - \frac{(w^j x)^2}{n^2} \right) = \frac{1}{(\pi x)^m} \prod_{j=0}^{m-1} \sin(w^j \pi x). \end{aligned}$$

Now we express the product of sine functions into a linear combinations of sine functions as

$$\prod_{j=0}^{m-1} \sin(w^j \pi x) = \frac{(-1)^{(m-1)/2}}{2^{m-1}} \sum_{\substack{\varepsilon_j = \pm 1 \\ 1 \leq j \leq m-1}} \text{sign}(\varepsilon) \cdot \sin(\pi x (w^{m-1} + \varepsilon_1 w^{m-2} + \dots + \varepsilon_{m-1}))$$

with

$$\text{sign}(\varepsilon) = \begin{cases} 1, & \text{if the number of } -1 \text{ in } \varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}) \text{ is even;} \\ -1, & \text{otherwise.} \end{cases}$$

As $w^m = 1$ and $m \geq 3$,

$$w^{m-1} + w^{m-2} + \cdots + w + 1 = 0.$$

So we are able to express the product as

$$\frac{(-1)^{(m-1)/2}}{2^{m-1}} \sum_{r=1}^{(m-1)/2} (-1)^{r-1} \sum_{0 \leq j_1 < j_2 < \cdots < j_r \leq m-1} \sin(2\pi x(w^{j_1} + w^{j_2} + \cdots + w^{j_r})).$$

The coefficient of x^{2mn+m} of the above combination of sine function gives the evaluation of $\zeta(\{2m\}^n)$. \square

Here are some evaluations of $\zeta(\{2m\}^n)$:

$$\zeta(\{6\}^n) = \frac{6 \cdot (2\pi)^{6n}}{(6n+3)!}, \quad (6)$$

$$\zeta(\{10\}^n) = \frac{10 \cdot (2\pi)^{10n}}{(10n+5)!} \left\{ 1 - \left(2 \cos \frac{2\pi}{5} \right)^{10n+5} - \left(2 \cos \frac{4\pi}{5} \right)^{10n+5} \right\}, \quad (7)$$

$$\begin{aligned} \zeta(\{14\}^n) = & \frac{14 \cdot (2\pi)^{14n}}{(14n+7)!} \left\{ 1 - \sum_{j=1}^3 \left(2 \cos \frac{2j\pi}{7} \right)^{14n+7} + \sum_{j=1}^3 \left(1 + 2 \cos \frac{2j\pi}{7} \right)^{14n+7} \right. \\ & \left. + \left(\frac{-1+i\sqrt{7}}{2} \right)^{14n+7} + \left(\frac{-1-i\sqrt{7}}{2} \right)^{14n+7} \right\}. \end{aligned} \quad (8)$$

Theorem 2. Suppose that m and n are positive integers with m even. Then

$$\zeta(\{2m\}^n) = \frac{(-1)^{n+1+\frac{m}{2}} \pi^{2mn}}{2^{m-2}(2mn+m)!} \cdot \Im \left(\sum_{\substack{\text{sign}(\varepsilon)=1 \\ \varepsilon_j=\pm 1, 1 \leq j \leq m-1}} (w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})^{2mn+m} \right)$$

with $w = \exp(2\pi i/2m)$ and

$$\text{sign}(\varepsilon) = \begin{cases} 1, & \text{if the number of } -1 \text{ in } \varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}) \text{ is even;} \\ -1, & \text{otherwise.} \end{cases}$$

Proof. The infinite product formula of the sine function implies that for $|x| < 1$ and $x \neq 0$,

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} (-1)^n \zeta(\{2m\}^n) x^{2mn} &= \prod_{n=1}^{\infty} \left(1 - \frac{x^{2m}}{n^{2m}} \right) = \prod_{n=1}^{\infty} \prod_{j=0}^{m-1} \left(1 - \frac{(w^j x)^2}{n^2} \right) \\ &= \frac{w^{-m(m-1)/2}}{\pi^m x^m} \prod_{j=0}^{m-1} \sin(w^j \pi x). \end{aligned} \quad (9)$$

Now the product of sine function can be expressed as

$$\begin{aligned} & \frac{(-1)^{m/2}}{2^{m-1}} \sum_{\substack{\varepsilon_j = \pm 1 \\ 1 \leq j \leq m-1}} \text{sign}(\varepsilon) \cdot \cos(\pi x(w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})) \\ &= \frac{(-1)^{m/2}}{2^{m-1}} \sum_{\substack{\varepsilon_j = \pm 1 \\ 1 \leq j \leq m-1}} \text{sign}(\varepsilon) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} [\pi x(w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})]^{2n}. \end{aligned}$$

The coefficient of x^{2mn+m} of Eq. (9) then gives the evaluation of $\zeta(\{2m\}^n)$ up to a constant.

$$\zeta(\{2m\}^n) = \frac{(-1)^n w^{-m(m-1)/2} \pi^{2mn}}{2^{m-1} (2mn+m)!} \sum_{\substack{\varepsilon_j = \pm 1 \\ 1 \leq j \leq m-1}} \text{sign}(\varepsilon) \cdot (w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})^{2mn+m}. \quad (10)$$

Let

$$\begin{aligned} A &= \{w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1} : \text{sign}(\varepsilon) = 1\}, \\ B &= \{w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1} : \text{sign}(\varepsilon) = -1\}. \end{aligned}$$

Choose $B_j \in B$ and write $B_j = w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1}$. If $\varepsilon_{m-2} = 1$, then

$$\begin{aligned} -\overline{B_j} &= w^m(w^{-m+1} + \varepsilon_1 w^{-m+2} + \cdots + \varepsilon_{m-3} w^{-2} + w^{-1} + \varepsilon_{m-1}) \\ &= w^{m-1} + \varepsilon_{m-3} w^{m-2} + \cdots + \varepsilon_1 w^2 + w - \varepsilon_{m-1}. \end{aligned}$$

Since the number of -1 in $(\varepsilon_{m-3}, \varepsilon_{m-4}, \dots, \varepsilon_1, 1, -\varepsilon_{m-1})$ is even, this implies $-\overline{B_j} \in A$. If $\varepsilon_{m-2} = -1$, then

$$\begin{aligned} \overline{B_j} &= w^{-m+1} + \varepsilon_1 w^{-m+2} + \cdots + \varepsilon_{m-3} w^{-2} - w^{-1} + \varepsilon_{m-1} \\ &= w^{m-1} - \varepsilon_{m-3} w^{m-2} - \cdots - \varepsilon_1 w^2 - w + \varepsilon_{m-1}. \end{aligned}$$

Since the number of -1 in $(-\varepsilon_{m-3}, \dots, -\varepsilon_1, -1, \varepsilon_{m-1})$ is even, $\overline{B_j} \in A$. The above fact make a one-to-one corresponding from B to A . Also we can simplify the summation in Eq. (10) if we write $A = \{A_1, A_2, \dots, A_{2^{m-2}}\}$ and $B = \{B_1, B_2, \dots, B_{2^{m-2}}\}$:

$$\begin{aligned} & \sum_{\substack{\varepsilon_j = \pm 1 \\ 1 \leq j \leq m-1}} \text{sign}(\varepsilon) \cdot (w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})^{2mn+m} \\ &= \sum_{j=1}^{2^{m-2}} (A_j^{2mn+m} - B_j^{2mn+m}) = \sum_{j=1}^{2^{m-2}} (A_j^{2mn+m} - \overline{A_j}^{2mn+m}) \\ &= 2i \Im \left(\sum_{\substack{\text{sign}(\varepsilon)=1 \\ \varepsilon_j = \pm 1, 1 \leq j \leq m-1}} (w^{m-1} + \varepsilon_1 w^{m-2} + \cdots + \varepsilon_{m-1})^{2mn+m} \right). \end{aligned}$$

Now $w^{-m(m-1)/2} \cdot i = (-1)^{1+\frac{m}{2}}$, this completes our proof. \square

Here are some explicit evaluations:

$$\zeta(\{4\}^n) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!}, \quad (11)$$

$$\zeta(\{8\}^n) = \frac{2^{4n+1}\pi^{8n}}{(8n+4)!} \left\{ \left(2 \cos \frac{\pi}{8}\right)^{8n+4} + \left(2 \sin \frac{\pi}{8}\right)^{8n+4} \right\}, \quad (12)$$

$$\zeta(\{12\}^n) = \frac{3 \cdot 2^{6n-1}\pi^{12n}}{(12n+6)!} \left\{ 2^{12n+6} + (\sqrt{3}+1)^{12n+6} + (\sqrt{3}-1)^{12n+6} \right\}. \quad (13)$$

4 Evaluations of $\zeta^*(\{2m\}^n)$

There are several evaluations of $\zeta^*(\{2m\}^n)$ available such as Hoffman [10] or S. Muneta [13, Theorem A], the later is in terms of Bernoulli numbers as

$$\zeta^*(\{2m\}^n) = \pi^{2mn} \times \left\{ \sum_{\substack{k_0 + \dots + k_{m-1} = mn \\ k_i \geq 0}} (-1)^{m(n-1)} \left(\prod_{j=0}^{m-1} \frac{(2^{2k_j} - 2)B_{2k_j}}{(2k_j)!} \right) \exp \left(\frac{2\pi i}{m} \sum_{\ell=0}^{m-1} \ell \cdot k_\ell \right) \right\},$$

for all positive integers m and n . However, we note that there is a little difference when m is even or odd. Here we prove a slight different version in Bernoulli polynomials.

Theorem 3. *Suppose that m, n are positive integers with m even. Then*

$$\zeta^*(\{2m\}^n) = (2\pi)^{2mn} \sum_{|p|=mn} \left(\prod_{j=1}^m \frac{B_{2p_j}(\frac{1}{2})}{(2p_j)!} \right) \exp \left(\frac{2\pi i}{m} \sum_{\ell=1}^m (\ell-1)p_\ell \right).$$

Proof. Let $w = \exp(2\pi i/2m)$ be the $2m$ -th root of unity. The generating function of $\zeta^*(\{2m\}^n)$ is

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \zeta^*(\{2m\}^n) x^{2mn} &= \prod_{n=1}^{\infty} \left(1 - \frac{x^{2m}}{n^{2m}} \right)^{-1} \\ &= \prod_{n=1}^{\infty} \prod_{j=1}^m \left(1 - \frac{(w^{j-1}x)^2}{n^2} \right)^{-1} \\ &= \prod_{j=1}^m \frac{2\pi i (w^{j-1}x) e^{\pi i (w^{j-1}x)}}{e^{2\pi i (w^{j-1}x)} - 1}. \end{aligned}$$

The coefficient of x^{2mn} in the power series expansion of the above product is

$$\sum_{|p|=mn} \prod_{j=1}^m \frac{(2\pi i w^{j-1})^{2p_j}}{(2p_j)!} B_{2p_j} \left(\frac{1}{2} \right).$$

Then we write it as

$$(2\pi)^{2mn} \sum_{|\mathbf{p}|=mn} \left(\prod_{j=1}^m \frac{B_{2p_j}(\frac{1}{2})}{(2p_j)!} \right) \exp \left(\frac{2\pi i}{m} \sum_{\ell=1}^m (\ell-1)p_\ell \right).$$

□

Theorem 4. Suppose that m, n are positive integers with m odd and $m \geq 3$. Then

$$\zeta^*(\{2m\}^n) = (2\pi i)^{2mn} \sum_{|\mathbf{p}|=2mn} \left(\prod_{j=1}^m \frac{B_{p_j}}{(p_j)!} \right) \exp \left(\frac{2\pi i}{m} \sum_{\ell=1}^m (\ell-1)p_\ell \right).$$

Proof. Let $w = \exp(2\pi i/m)$ be the m -th root of unity. The generating function of $\zeta^*(\{2m\}^n)$ is

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \zeta^*(\{2m\}^n) x^{2mn} &= \prod_{n=1}^{\infty} \left(1 - \frac{x^{2m}}{n^{2m}} \right)^{-1} \\ &= \prod_{n=1}^{\infty} \prod_{j=1}^m \left(1 - \frac{(w^{j-1}x)^2}{n^2} \right)^{-1} \\ &= \prod_{j=1}^m \frac{2\pi i(w^{j-1}x)e^{\pi i(w^{j-1}x)}}{e^{2\pi i(w^{j-1}x)} - 1}. \end{aligned}$$

As $m \geq 3$,

$$1 + w + w^2 + \cdots + w^{m-1} = \frac{1 - w^m}{1 - w} = 0,$$

so the product is equal to

$$\prod_{j=1}^m \frac{2\pi i(w^{j-1}x)}{e^{2\pi i(w^{j-1}x)} - 1}.$$

The coefficient of x^{2mn} in the power series expansion at $x = 0$ of the above product is

$$\sum_{|\mathbf{p}|=2mn} \prod_{j=1}^m \frac{(2\pi i w^{j-1})^{p_j}}{(p_j)!} B_{p_j}.$$

It is also written as

$$(2\pi i)^{2mn} \sum_{|\mathbf{p}|=2mn} \left(\prod_{j=1}^m \frac{B_{p_j}}{(p_j)!} \right) \exp \left(\frac{2\pi i}{m} \sum_{\ell=1}^m (\ell-1)p_\ell \right).$$

This proves our theorem. □

Remark. During the proof of the above theorem, if we stop at the step

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^{2m}}{n^{2m}} \right)^{-1} = \prod_{j=1}^m \frac{2\pi i(w^{j-1}x)e^{\pi i(w^{j-1}x)}}{e^{2\pi i(w^{j-1}x)} - 1}$$

and then expand into power series, we obtain another expression of $\zeta^*(\{2m\}^n)$ with m odd:

$$\zeta^*(\{2m\}^n) = (\pi i)^{2mn} \sum_{|\mathbf{q}|=mn} \left(\prod_{j=1}^m \frac{2^{2q_j}}{(2q_j)!} B_{2q_j} \left(\frac{1}{2} \right) \right) \exp \left(\frac{4\pi i}{m} \sum_{\ell=1}^m (\ell-1)q_\ell \right). \quad (14)$$

Here are some explicit evaluations:

$$\zeta^*(\{4\}^n) = (2\pi)^{4n} \sum_{|\mathbf{q}|=2n} (-1)^{q_2} \frac{B_{2q_1} \left(\frac{1}{2} \right) B_{2q_2} \left(\frac{1}{2} \right)}{(2q_1)!(2q_2)!}, \quad (15)$$

$$\zeta^*(\{6\}^n) = (-1)^n (2\pi)^{6n} \sum_{|\mathbf{q}|=6n} \frac{B_{q_1} B_{q_2} B_{q_3}}{q_1! q_2! q_3!} \exp \left(\frac{2\pi i}{3} (q_2 + 2q_3) \right), \quad (16)$$

$$\zeta^*(\{8\}^n) = (2\pi)^{8n} \sum_{|\mathbf{q}|=4n} \left(\prod_{j=1}^4 \frac{B_{2q_j} \left(\frac{1}{2} \right)}{(2p_j)!} \right) \exp \left(\frac{\pi i}{2} \sum_{\ell=1}^4 (\ell-1)q_\ell \right). \quad (17)$$

The evaluation of $E(4n, k)$ was conjectured in [9] as

$$(-1)^k \frac{(-4\pi^4)^n}{(4n+2)!} \sum_{u=0}^{n-k} \binom{n-u}{k} \binom{4n+2}{4u} Y_u,$$

where

$$Y_u = \frac{2}{(-4)^u} \sum_{u_1=0}^{2u} (-1)^{u_1} \binom{4u}{2u_1} (2-4^{u_1}) B_{2u_1} (2-4^{2u-u_1}) B_{2(2u-u_1)}.$$

Now it becomes our Theorem C in a slight different notation.

Theorem C. *For a pair of positive integers n, k with $1 \leq k \leq n$, we have*

$$\begin{aligned} E(4n, k) &= \frac{2^{2n+1} \pi^{4n}}{(4n+2)!} \sum_{q=0}^{n-k} (-1)^{n-q-k} \binom{n-q}{k} \binom{4n+2}{4q} \sum_{|\mathbf{q}|=2q} \binom{4q}{2q_1} (-1)^{q_1} 2^{2q} B_{2q_1} \left(\frac{1}{2} \right) B_{2q_2} \left(\frac{1}{2} \right). \end{aligned}$$

Proof. Substituting $m = 4$ in the formula of $E(mn, k)$ in Theorem A, we have

$$E(4n, k) = \sum_{p+q=n} (-1)^{p-k} \binom{p}{k} \zeta(\{4\}^p) \zeta^*(\{4\}^q).$$

By Eq. (11) and Eq. (15), $E(4n, k)$ becomes

$$\sum_{p+q=n} (-1)^{p-k} \binom{p}{k} \frac{2^{2p+1} \pi^{4p}}{(4p+2)!} \sum_{|\mathbf{q}|=2q} \frac{(-1)^{q_1} (2\pi)^{4q}}{(2q_1)!(2q_2)!} B_{2q_1} \left(\frac{1}{2} \right) B_{2q_2} \left(\frac{1}{2} \right)$$

Note that the summation is equal to zero if $p < k$, so we can rewrite it as the form stated in the theorem. \square

Based on the evaluations of $\zeta(\{6\}^p)$ and $\zeta^*(\{6\}^q)$ (see Eq. (6) and Eq. (16)), we have the following evaluation of $E(6n, k)$:

$$E(6n, k) = \frac{(-1)^{n-k} 6(2\pi)^{6n}}{(6n+3)!} \sum_{q=0}^{n-k} \binom{n-q}{k} \binom{6n+3}{6q} \\ \times \sum_{|\mathbf{q}|=6q} \frac{\exp(2\pi i(q_2 + 2q_3)/3)(6q)!}{(q_1)!(q_2)!(q_3)!} B_{q_1} B_{q_2} B_{q_3},$$

or using Eq.(14) we have another expression

$$E(6n, k) = \sum_{|\boldsymbol{\alpha}|=n} \zeta(6\alpha_1, 6\alpha_2, \dots, 6\alpha_k) \\ = \frac{(-1)^{n-k} 6(2\pi)^{6n}}{(6n+3)!} \sum_{q=0}^{n-k} \binom{n-q}{k} \binom{6n+3}{6q} \\ \times \sum_{|\mathbf{q}|=3q} \frac{\exp(4\pi i(q_2 + 2q_3)/3)(6q)!}{(2q_1)!(2q_2)!(2q_3)!} B_{2q_1} \left(\frac{1}{2}\right) B_{2q_2} \left(\frac{1}{2}\right) B_{2q_3} \left(\frac{1}{2}\right).$$

At last we give the evaluation of $E(8n, k)$:

$$E(8n, k) = \sum_{|\boldsymbol{\alpha}|=n} \zeta(8\alpha_1, 8\alpha_2, \dots, 8\alpha_k) \\ = \frac{2^{4n+1} \pi^{8n}}{(8n+4)!} \sum_{q=0}^{n-k} (-1)^{n-q-k} \binom{n-q}{k} \binom{8n+4}{8q} \left\{ \left(2 \cos \frac{\pi}{8}\right)^{8n-8q+4} + \left(2 \sin \frac{\pi}{8}\right)^{8n-8q+4} \right\} \\ \times \sum_{|\mathbf{q}|=4q} 2^{4q} (8q)! \left(\prod_{j=1}^4 \frac{B_{2q_j}(\frac{1}{2})}{(2q_j)!} \right) \exp \left(\frac{\pi i}{2} \sum_{\ell=1}^4 (\ell-1) q_\ell \right).$$

5 A final remark

According to our Theorem A, the evaluation of $E(mn, k)$ depends on $\zeta(\{m\}^p)$ and $\zeta^*(\{m\}^q)$ with $p+q = n$. Both multiple zeta values can be evaluated in terms of $\zeta(m), \zeta(2m), \dots, \zeta(nm)$.

Here we list some preliminaries about symmetric functions. For more details, we refer the reader to [12, 15]. Let e_m , h_m , and p_m be the m -th elementary, complete homogeneous, and power-sum symmetric polynomials, in infinitely many variables x_1, x_2, \dots , respectively.

They have associated generating functions

$$\begin{aligned} E(t) &:= \sum_{j=0}^{\infty} e_j t^j = \prod_{i=1}^{\infty} (1 + tx_i), \\ H(t) &:= \sum_{j=0}^{\infty} h_j t^j = \prod_{i=1}^{\infty} \frac{1}{1 - tx_i} = E(-t)^{-1}, \\ P(t) &:= \sum_{j=1}^{\infty} p_j t^{j-1} = \sum_{i=1}^{\infty} \frac{x_i}{1 - tx_i} = \frac{H'(t)}{H(t)} = \frac{E'(-t)}{E(-t)}. \end{aligned}$$

Let the modified Bell polynomials $P_m(x_1, x_2, \dots, x_m)$ be defined by [4, 5]

$$\exp\left(\sum_{k=1}^{\infty} \frac{x_k}{k} z^k\right) = \sum_{m=0}^{\infty} P_m(x_1, x_2, \dots, x_m) z^m.$$

The general explicit expression for P_m is

$$P_m(x_1, \dots, x_m) = \sum_{k_1+2k_2+\dots+mk_m=m} \frac{1}{k_1! \cdots k_m!} \left(\frac{x_1}{1}\right)^{k_1} \cdots \left(\frac{x_m}{m}\right)^{k_m}.$$

Lemma 1. *Let j be a nonnegative integer. Then we have*

$$e_j = P_j(p_1, -p_2, \dots, (-1)^{j+1} p_j), \quad (18)$$

$$h_j = P_j(p_1, p_2, \dots, p_j). \quad (19)$$

Proof. We begin with the generating function of e_j .

$$\begin{aligned} \sum_{j=0}^{\infty} e_j t^j &= \prod_{i=1}^{\infty} (1 + tx_i) = \exp\left[\sum_{i=1}^{\infty} \log(1 + tx_i)\right] \\ &= \exp\left[\sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{t^\ell}{\ell} \sum_{i=1}^{\infty} x_i^\ell\right] \\ &= \sum_{m=0}^{\infty} P_m(p_1, -p_2, \dots, (-1)^{m+1} p_m) t^m. \end{aligned}$$

This gives us Eq. (18). Eq. (19) can be easily derived by a similar method, hence we omit it. \square

Let $x_k = k^{-s}$, for all $k \geq 1$. Then

$$e_n = \zeta(\{s\}^n), \quad h_n = \zeta^*(\{s\}^n), \quad \text{and} \quad p_n = \zeta(ns).$$

Note that we denote that $\zeta(\{s\}^0) = \zeta^*(\{s\}^0) = 1$. Now Eq. (18) and Eq. (19) give us the evaluations of $\zeta(\{s\}^n)$ and $\zeta^*(\{s\}^n)$:

$$\zeta(\{s\}^n) = P_n(\zeta(s), -\zeta(2s), \dots, (-1)^{n+1} \zeta(ns)), \quad (20)$$

$$\zeta^*(\{s\}^n) = P_n(\zeta(s), \zeta(2s), \dots, \zeta(ns)). \quad (21)$$

For example, we have

$$\begin{aligned}\zeta(\{s\}^5) &= \frac{1}{5!} (\zeta(s)^5 - 10\zeta(s)^3\zeta(2s) + 20\zeta(s)^2\zeta(3s) - 30\zeta(s)\zeta(4s) \\ &\quad + 15\zeta(s)\zeta(2s)^2 - 20\zeta(2s)\zeta(3s) + 24\zeta(5s)) ; \\ \zeta^*(\{s\}^5) &= \frac{1}{5!} (\zeta(s)^5 + 10\zeta(s)^3\zeta(2s) + 20\zeta(s)^2\zeta(3s) + 30\zeta(s)\zeta(4s) \\ &\quad + 15\zeta(s)\zeta(2s)^2 + 20\zeta(2s)\zeta(3s) + 24\zeta(5s)) .\end{aligned}$$

We can give another evaluations of $\zeta(\{2m\}^n)$ and $\zeta^*(\{2m\}^n)$ in terms of Bernoulli numbers.

Proposition 3. *Let m, n be positive integers. Then we have*

$$\zeta(\{2m\}^n) = (-1)^{n(m+1)} (2\pi)^{2mn} \sum_{\substack{a_1+2a_2+\dots+na_n=n \\ a_i \geq 0}} \prod_{k=1}^n \frac{1}{a_k!} \left(\frac{B_{2km}}{2k \cdot (2km)!} \right)^{a_k}$$

and

$$\zeta^*(\{2m\}^n) = (-1)^{mn} \cdot (2\pi)^{2mn} \sum_{\substack{a_1+2a_2+\dots+na_n=n \\ a_i \geq 0}} \prod_{k=1}^m \frac{(-1)^{a_k}}{a_k!} \left(\frac{B_{2km}}{2k \cdot (2km)!} \right)^{a_k} .$$

Note that Eq. (20) and Eq. (21) could be obtained by setting $\mathbf{i} = (s, s, \dots, s)$ in Eq. (1) and Eq. (2), respectively.

Now we can refined Theorem A as follows.

Theorem 5. *For a pair of positive integers m, n , with $m \geq 2$, we have for $1 \leq k \leq n$ that*

$$\begin{aligned}E(mn, k) &= \sum_{p+q=n} (-1)^{p-k} \binom{p}{k} \\ &\quad \times P_p(\zeta(m), -\zeta(2m), \dots, (-1)^{p+1}\zeta(pm)) P_q(\zeta(m), \zeta(2m), \dots, \zeta(qm)).\end{aligned}$$

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